MST2241806F1PV1

MST224/C

Module Examination 2018

Mathematical methods

Wednesday 6 June 2018 **10.00 am – 1.00 pm**

Time allowed: 3 hours

You are **not** allowed to use a calculator in this examination.

There are THREE parts to this paper, each taking approximately one hour.

In each part of the paper the questions are arranged, as far as possible, in the order in which they appear in the module.

Part 1 consists of 11 computer marked questions, each worth 3 marks. Mark your answers on the form provided using an HB pencil. Detailed instructions on filling in the computer marked examination (CME) form are given overleaf.

Part 2 consists of 7 questions, each worth 5 marks.

Part 3 consists of 4 questions, each worth 16 marks. The marks from your best two answers will be added together to give a maximum of 32 marks for this part.

In Parts 2 and 3: Write your answers in **pen** in the answer book(s) provided. The marks allocated to each part of each question are given in square brackets in the margin. Unless you are directed otherwise in the question, you may use any formula or other information from the Handbook in your answers. Do **not** cross out any answers unless you have a better alternative answer – everything not crossed out may receive credit.

At the end of the examination: Check that you have written your personal identifier and examination number on each answer book used, and that you have written your personal identifier and name on the CME form. **Failure to do so may mean that your work cannot be identified.** Attach your signed desk record to the front of your answer book(s) using the round paper fastener, then attach the CME form and your question paper to the back of the answer book(s) using the flat paperclip.

Instructions for filling in the computer marked examination (CME) form

- You will find one CME form provided with this paper. The invigilator has a supply of spare forms if you should need any.
- You should use an **HB pencil** to make entries on the CME form. If you make any smudges or other spurious marks on the form that you cannot cancel out clearly, you should *ask the invigilator for a new form*, and transfer your entries to it.
- On Part 1 of the CME form, you must *write* in your personal identifier (NOT the examination number) and the 'assignment number' for this examination (**MST224 81**). You should also *pencil* across the cells in the two blocks in Part 1 of the form corresponding to your personal identifier and the assignment number given above. On the sample CME form opposite Part 1 has been completed for a fictitious student so that you can see how to complete this Part of the form. We suggest you check that Part 1 on the CME form has been completed correctly before moving on to Part 2 of the examination paper.
- For each question, you **must** pencil across **either** the required number of answer cells **or** the 'don't know' cell.
- If you do not wish to answer a question, pencil across the 'don't know' cell ('?').
- If you think that a question is unsound in any way, pencil across the 'unsound' cell ('U'), *in addition* to pencilling across either an answer cell or the 'don't know' cell.
- We suggest that in the first instance, you answer by pencilling across the relevant cells on the sample CME form opposite. Check your answers before transferring them to your CME form.
- You should note that **no additional time** will be allowed at the end of the three-hour period for transferring your marks to the CME form.
- Failure to follow the above instructions may mean that we will not be able to award you a mark for this part of the examination.
- Use the answer book(s) provided for any rough work.
- Incorrect answers will not be penalised, i.e. you will not lose marks for incorrect answers.

COMPUTER MARKED EXAMINATION FORM

08-10

PART 1

PERSONAL IDENTIFIER
(NOT Examination Number)

A1234567

ASSIGNMENT NUMBER
(as given on the question paper)

MST224 81

NAME and INITIALS

A. STUDENT

IMPORTANT NOTES

1. Please check that all of this form has been completed including the coloured boxes in Part 1 with your Personal Identifier, Module and Assignment Number.
2. Only use an HB pencil to complete this form and press firmly.
3. Shown opposite are examples of the correct marks to use. Do not use any other types of mark.

PLEASE DO NOT PUNCH HOLES IN THIS FORM

Fold this form INWARDS along the dotted line

PERSONAL IDENTIFIER

A	0	0	0	0	0	0	0
B	1	1	1	1	1	1	1
C	2	2	2	2	2	2	2
D	3	3	3	3	3	3	3
E	4	4	4	4	4	4	4
F	5	5	5	5	5	5	5
G	6	6	6	6	6	6	6
H	7	7	7	7	7	7	7
K	8	8	8	8	8	8	8
L	9	9	9	9	9	9	9
M							X
N							
P							
R							
S							
T							
U							
V							
W							
X							
Y							

CORRECT MARK

Pencil across between dots

CANCELLATION MARK

To cancel a mark pencil in the coloured part of the cell. Do not rub out

MODULE AND ASSIGNMENT NUMBER

A	A	A	A	0	0	0	0	0
B	B	B	B	1	1	1	1	1
C	C	C	C	2	2	2	2	2
D	D	D	D	3	3	3	3	3
E	E	E	E	4	4	4	4	4
F	F	F	F	5	5	5	5	5
G	G	G	G	6	6	6	6	6
H	H	H	H	7	7	7	7	7
K	K	K	K	8	8	8	8	8
L	L	L	L	9	9	9	9	9
M	M	M	M					
N	N	N	N					
P	P	P	P					
R	R	R	R					
S	S	S	S					
T	T	T	T					
U	U	U	U					
W	W	W	W					
X	X	X	X					
Y	Y	Y	Y					
Z	Z	Z	Z					

OFFICE USE ONLY

0	0
1	1
2	2
3	3
4	4
5	5
6	6
7	7
8	8
9	9

OFFICE USE ONLY

The Open University

PART 2

	ANSWER											ANSWER											ANSWER									
1	A	B	C	D	E	F	G	H	?	U	21	A	B	C	D	E	F	G	H	?	U	41	A	B	C	D	E	F	G	H	?	U
2	A	B	C	D	E	F	G	H	?	U	22	A	B	C	D	E	F	G	H	?	U	42	A	B	C	D	E	F	G	H	?	U
3	A	B	C	D	E	F	G	H	?	U	23	A	B	C	D	E	F	G	H	?	U	43	A	B	C	D	E	F	G	H	?	U
4	A	B	C	D	E	F	G	H	?	U	24	A	B	C	D	E	F	G	H	?	U	44	A	B	C	D	E	F	G	H	?	U
5	A	B	C	D	E	F	G	H	?	U	25	A	B	C	D	E	F	G	H	?	U	45	A	B	C	D	E	F	G	H	?	U
6	A	B	C	D	E	F	G	H	?	U	26	A	B	C	D	E	F	G	H	?	U	46	A	B	C	D	E	F	G	H	?	U
7	A	B	C	D	E	F	G	H	?	U	27	A	B	C	D	E	F	G	H	?	U	47	A	B	C	D	E	F	G	H	?	U
8	A	B	C	D	E	F	G	H	?	U	28	A	B	C	D	E	F	G	H	?	U	48	A	B	C	D	E	F	G	H	?	U
9	A	B	C	D	E	F	G	H	?	U	29	A	B	C	D	E	F	G	H	?	U	49	A	B	C	D	E	F	G	H	?	U
10	A	B	C	D	E	F	G	H	?	U	30	A	B	C	D	E	F	G	H	?	U	50	A	B	C	D	E	F	G	H	?	U
11	A	B	C	D	E	F	G	H	?	U	31	A	B	C	D	E	F	G	H	?	U	51	A	B	C	D	E	F	G	H	?	U
12	A	B	C	D	E	F	G	H	?	U	32	A	B	C	D	E	F	G	H	?	U	52	A	B	C	D	E	F	G	H	?	U
13	A	B	C	D	E	F	G	H	?	U	33	A	B	C	D	E	F	G	H	?	U	53	A	B	C	D	E	F	G	H	?	U
14	A	B	C	D	E	F	G	H	?	U	34	A	B	C	D	E	F	G	H	?	U	54	A	B	C	D	E	F	G	H	?	U
15	A	B	C	D	E	F	G	H	?	U	35	A	B	C	D	E	F	G	H	?	U	55	A	B	C	D	E	F	G	H	?	U
16	A	B	C	D	E	F	G	H	?	U	36	A	B	C	D	E	F	G	H	?	U	56	A	B	C	D	E	F	G	H	?	U
17	A	B	C	D	E	F	G	H	?	U	37	A	B	C	D	E	F	G	H	?	U	57	A	B	C	D	E	F	G	H	?	U
18	A	B	C	D	E	F	G	H	?	U	38	A	B	C	D	E	F	G	H	?	U	58	A	B	C	D	E	F	G	H	?	U
19	A	B	C	D	E	F	G	H	?	U	39	A	B	C	D	E	F	G	H	?	U	59	A	B	C	D	E	F	G	H	?	U
20	A	B	C	D	E	F	G	H	?	U	40	A	B	C	D	E	F	G	H	?	U	60	A	B	C	D	E	F	G	H	?	U

Part 1

You should attempt **all** questions in this part of the paper. Each question is worth 3 marks. You should answer in **pencil** on the CME form provided.

Question 1

Given that

$$\frac{y}{y-3} = 3x+5,$$

select the option that gives y as a function of x.

Options

A $y = \dfrac{3x+5}{3x+4}$

B $y = \dfrac{3x+4}{9x+15}$

C $y = 3\dfrac{3x+5}{3x+4}$

D $y = \dfrac{3x+4}{3x+5}$

Question 2

Select the option that gives an intermediate step in applying the method of integration by parts to the integral

$$\int x^2 e^{-x}\, dx.$$

Options

A $\displaystyle\int x^2 e^{-x}\, dx = x^2 e^{-x} + \int \left(-2xe^{-x}\right)\, dx$

B $\displaystyle\int x^2 e^{-x}\, dx = -x^2 e^{-x} - \int \left(-2xe^{-x}\right)\, dx$

C $\displaystyle\int x^2 e^{-x}\, dx = -x^2 e^{-x} + \int \left(-2xe^{-x}\right)\, dx$

D $\displaystyle\int x^2 e^{-x}\, dx = -2xe^{-x} - \int \left(-x^2 e^{-x}\right)\, dx$

Question 3

Select the option that gives the general solution of the differential equation

$$\frac{dy}{dx} = \frac{1}{y\,(x^2+1)},$$

where in the options C is an arbitrary constant.

Options

A $y = C + \arctan x$

B $y = e^{C+\arctan x}$

C $y = \pm\sqrt{2C + 2\arctan x}$

D $y = \pm\frac{1}{3}\sqrt{18C + 6x^3 + 18x}$

Question 4

Consider the matrix

$$\mathbf{A} = \begin{bmatrix} 0 & 2 & -2 \\ 1 & 1 & -2 \\ 3 & 0 & 2 \end{bmatrix}.$$

Select the option that gives the determinant of $\mathbf{A}$.

Options

A -10 **B** -22 **C** 10 **D** 22

Question 5

Consider using the method of Gaussian elimination to solve the following system of linear equations:

$$\begin{bmatrix} -2 & 8 & 2 \\ -6 & 4 & 7 \\ 3 & 9 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}.$$

Select the option that gives the first step of the Gaussian elimination procedure.

Options

A $\mathbf{R}_{2\text{a}} = \mathbf{R}_2 - 2\mathbf{R}_1$ **B** $\mathbf{R}_{2\text{a}} = \mathbf{R}_2 + 2\mathbf{R}_1$

C $\mathbf{R}_{2\text{a}} = \mathbf{R}_2 + 3\mathbf{R}_1$ **D** $\mathbf{R}_{2\text{a}} = \mathbf{R}_2 - 3\mathbf{R}_1$

Question 6

The system of differential equations

$$\begin{cases} \dot{x} = x + 2y - 10e^{-t}, \\ \dot{y} = 4x + y - 14e^{-t}, \end{cases}$$

has complementary function

$$\begin{bmatrix} x \\ y \end{bmatrix} = \alpha \begin{bmatrix} \sqrt{2} \\ 2 \end{bmatrix} \exp\left((1 + 2\sqrt{2})t\right) + \beta \begin{bmatrix} \sqrt{2} \\ -2 \end{bmatrix} \exp\left((1 - 2\sqrt{2})t\right).$$

Select the option that gives the particular integral.

Options

A $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3e^{-t} \\ -2e^{-t} \end{bmatrix}$ **B** $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3e^{-t} \\ 2e^{-t} \end{bmatrix}$

C $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -3e^{-t} \\ 2e^{-t} \end{bmatrix}$ **D** $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2e^{-t} \\ 3e^{-t} \end{bmatrix}$

Question 7

Select the option that gives the value of the integral

$$\int_{y=0}^{y=2}\left(\int_{x=0}^{x=1}(3x^2+xy)\,dx\right)dy.$$

Options

A 3	**B** -3	**C** 8	**D** -8

Question 8

A vector field is defined in cylindrical coordinates by

$$\mathbf{F}(r,\theta,\phi) = r\,\mathbf{e}_r - r^2\sin\phi\,\mathbf{e}_\phi + r\,\mathbf{e}_z.$$

Select the option that gives the divergence of $\mathbf{F}$.

Options

A $\nabla\cdot\mathbf{F} = -r\cos\phi + 1$

B $\nabla\cdot\mathbf{F} = r\,(-\sin\phi + 2)$

C $\nabla\cdot\mathbf{F} = -r\cos\phi + 2$

D $\nabla\cdot\mathbf{F} = -r^2\cos\phi + 1$

Question 9

A conservative vector field is given by

$$\mathbf{F} = -\sin y\,\mathbf{i} - x\cos y\,\mathbf{j} + e^{-z}\,\mathbf{k}.$$

Select the option that gives a potential function for $\mathbf{F}$.

Options

A $U(x,y,z) = 2x\sin y + e^{-z}$ **B** $U(x,y,z) = -x\sin y - e^{-z}$

C $U(x,y,z) = x\sin y + e^{-z}$ **D** $U(x,y,z) = x\cos y + \sin y - e^{-z}$

Question 10

Consider the function $f(t) = 2t^3 + 4t$ defined on the interval $-1 \le t \le 1$.

Select the option that best describes this function.

Options

A The function is both even and odd.

B The function is neither even nor odd.

C The function is even.

D The function is odd.

Question 11

Consider the system of non-linear differential equations,

$$\frac{dx}{dt} = -x^2 + xy,$$

$$\frac{dy}{dt} = -x^2y^2 + x.$$

Select the option that gives the Jacobian matrix for this system.

Options

A $\begin{bmatrix} -2x+y & x \\ -2xy^2+1 & -2x^2y \end{bmatrix}$

B $\begin{bmatrix} -2xy^2+1 & -2x^2y \\ -2x+y & x \end{bmatrix}$

C $\begin{bmatrix} x & -2x+y \\ -2x^2y & -2xy^2+1 \end{bmatrix}$

D $\begin{bmatrix} -2x^2y & -2xy^2+1 \\ x & -2x+y \end{bmatrix}$

Part 2

You should attempt **all** questions in this part of the paper. Each question is worth 5 marks. You should answer in **pen** in the answer book(s) provided.

Question 12

Consider the differential equation

$$\frac{dy}{dx} = \frac{y}{x}, \quad x, y > 0.$$

(a) Find the general solution of the differential equation explicitly in the form $y = f(x)$. [3]

(b) Find the particular solution that satisfies $y(1) = 2$. [2]

Question 13

Consider the differential equation

$$\frac{d^2y}{dx^2} + 7\frac{dy}{dx} + 12y = 24x + 14.$$

Given that the complementary function is

$$y(x) = Ae^{-4x} + Be^{-3x},$$

find a particular integral. [5]

Question 14

Consider the vectors

$$\mathbf{a} = 2\mathbf{j} - 2\mathbf{k}, \quad \mathbf{b} = 3\mathbf{i} + 2\mathbf{j} + \mathbf{k}.$$

(a) Calculate the scalar product of **a** and **b** and hence calculate the cosine of the angle between **a** and **b**. [3]

(b) Calculate the vector product $\mathbf{a} \times \mathbf{b}$. [2]

Question 15

Consider the matrix

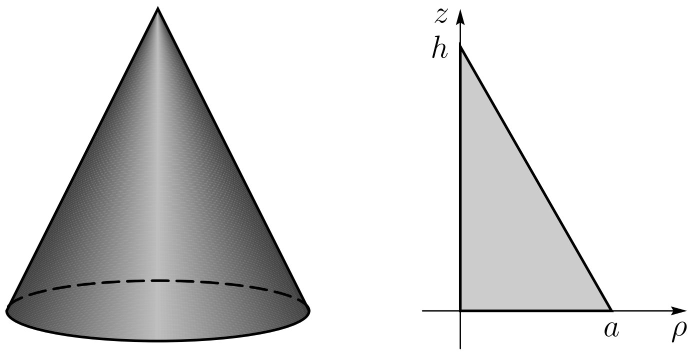
$$\mathbf{A} = \begin{bmatrix} 3 & -1 & -1 \\ -1 & 3 & 1 \\ -1 & 1 & 5 \end{bmatrix}.$$

(a) Given that the matrix **A** has eigenvector $\begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}$, calculate the corresponding eigenvalue. [2]

(b) Given that one of the eigenvalues of the matrix **A** is 3, calculate the corresponding eigenvector. [3]

Question 16

Consider the problem of calculating the volume of the cone shown in the left-hand diagram below.



As the cone is axially symmetric we use cylindrical coordinates with origin at the centre of the base and z oriented along the axis of symmetry. A section through the cone when $\phi = 0$ is shown in the right-hand diagram. The radius of the base is a and the height of the cone is h.

Write down an integral in cylindrical coordinates that would evaluate to the volume of the cone (you are not asked to evaluate this integral). *[5]*

Question 17

A vector field $\mathbf{v}$ is expressed in spherical coordinates as

$$\mathbf{v}(r, \theta, \phi) = \sin\phi \sin\theta \, \mathbf{e}_r + \sin\phi \cos\theta \, \mathbf{e}_\theta + \cos\phi \, \mathbf{e}_\phi.$$

Calculate $\boldsymbol{\nabla} \times \mathbf{v}$ and hence show that $\mathbf{v}$ is everywhere conservative. *[5]*

Question 18

The system of non-linear differential equations

$$\dot{x} = \sin x \cos^2 y,$$
$$\dot{y} = \sin x + \cos(\tfrac{1}{2}y),$$

has an equilibrium point at $(0, \pi)$.

(a) Calculate the Jacobian matrix of this system of equations and evaluate this matrix at the given equilibrium point. *[3]*

(b) Use your answer to part (a) to classify this equilibrium point. *[2]*

Part 3

You should attempt **two** questions in this part of the paper. Each question is worth 16 marks. All of your answers will be marked, and the marks from your best two answers will be added together, giving a maximum of 32 marks from this part. You should answer in **pen** in the answer book(s) provided.

Question 19

Find the general solution of the differential equation

$$\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + 5x = 8e^{-t},$$

and also the particular solution that satisfies the initial conditions $x(0) = 1$ and $\dot{x}(0) = 0$. *[16]*

Question 20

Find the general solution of the simultaneous linear differential equations

$$\frac{dx}{dt} = -5x - 2y + 5t - 1,$$

$$\frac{dy}{dt} = -x - 4y + t - 4.$$

[16]

Question 21

Consider the function

$$f(x, y) = x^2 + xy^2 - 6xy + 5x.$$

(a) Calculate the partial derivatives $f_x, f_y, f_{xx}, f_{xy}, f_{yy}$. *[5]*

(b) Find the stationary points of f. *[5]*

(c) Classify each of the stationary points. *[6]*

Question 22

Consider the partial differential equation and boundary conditions

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{D}\frac{\partial u}{\partial t} \quad (0 < x < L,\ t > 0),$$

$$u(0,t) = 0, \quad \frac{\partial u}{\partial x}(L,t) = 0, \quad (t > 0),$$

where D is a non-zero constant.

(a) Use separation of variables, with $u(x,t) = X(x)\,T(t)$, to show that

$$X''(x) = \mu X(x), \quad \text{for some constant } \mu.$$

Write down the corresponding differential equation for $T(t)$. [3]

(b) Write down the boundary conditions for $X(x)$ implied by the boundary conditions for $u(x,t)$. [3]

(c) Show that $X(x) = A\cos(kx) + B\sin(kx)$ where $k > 0$ is a solution of the differential equation given in part (a) and clearly state the relationship between k and μ. [2]

(d) Use the boundary conditions that you found in part (b) to determine the values that k takes for non-trivial solutions. [4]

(e) Adding the non-trivial solutions gives the following general solution, which you are not asked to derive:

$$u(x,t) = \sum_{n=1}^{\infty} C_n \exp\left(-\frac{D(2n-1)^2\pi^2 t}{4L^2}\right)\sin\left(\frac{(2n-1)\pi x}{2L}\right).$$

Find the particular solution corresponding to the initial condition

$$u(x,0) = 0.2\sin\left(\frac{5\pi x}{2L}\right) \quad (0 \le x \le L).$$ [4]

[END OF QUESTION PAPER]